

On perturbation of a surjective convolution operator

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Abstract: Let $\mu \in \mathcal{E}'(\mathbb{R}^n)$ be a compactly supported distribution such that its support is a convex set with non-empty interior. Let X_2 be a convex domain in \mathbb{R}^n , $X_1 = X_2 + \text{supp } \mu$. Assuming that a convolution operator $A : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$ acting by the rule $(Af)(x) = (\mu * f)(x)$ is surjective we provide a condition on a linear continuous operator $B : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$ that guarantees surjectivity of the operator $A + B$.

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1 Introduction

1.1. Statement of the problem and the main result. By $\mathcal{E}(X)$, where X is an open subset of \mathbb{R}^n , we mean the space of infinitely differentiable functions in X with the topology defined by the system of semi-norms

$$\|f\|_{K,N} = \sup_{x \in K, |\alpha| \leq N} |(D^\alpha f)(x)|, \quad K \Subset X, N \in \mathbb{Z}_+.$$

Its strong dual space $\mathcal{E}'(X)$ is the space of distributions with compact support in X .

If $0 \neq \mu \in \mathcal{E}'(\mathbb{R}^n)$ and X_1, X_2 are two non-empty open sets in \mathbb{R}^n such that

$$X_2 + \text{supp } \mu \subset X_1, \tag{1}$$

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then the convolution $\mu * f$ of distribution μ and a function $f \in \mathcal{E}(X_1)$ defined by the rule

$$(\mu * f)(x) = \mu(f(x + y)), \quad x \in X_2,$$

is in $\mathcal{E}(X_2)$.

The problem of characterizing the surjectivity of convolution operators and, in particular, of partial differential operators, has interested several authors. L. Ehrenpreis [1] and B. Malgrange [2] established that for each nonzero polynomial P of n variables $P(D)(C^\infty(\mathbb{R}^n)) = C^\infty(\mathbb{R}^n)$. For a distribution $\mu \in \mathcal{E}'(\mathbb{R}^n)$, $\mu \neq 0$, L. Ehrenpreis [3] proved that the convolution operator $f \rightarrow \mu * f$ acting from $C^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$ is surjective if and only if μ is invertible, respectively, if its Fourier-Laplace transform $\hat{\mu}$ defined by

$$\hat{\mu}(z) = \mu(e^{\langle -iz, \xi \rangle}), \quad z \in \mathbb{C}^n,$$

is slowly decreasing, that is, if there exists a constant $a > 0$ such that for every $\xi \in \mathbb{R}^n$ there exists $\eta \in \mathbb{R}^n$ satisfying

$$\|\xi - \eta\| \leq a \ln(2 + \|\xi\|)$$

and

$$|\hat{\mu}(\eta)| \geq (a + \|\xi\|)^{-a}.$$

The answer to a question concerning the surjectivity of convolution operators in general case was given by L. Hörmander [4]-[6]. He proved that the convolution equation $\mu * f = g$ has a solution $f \in \mathcal{E}(X_1)$ for every $g \in \mathcal{E}(X_2)$ if and only if μ is invertible and the pair (X_1, X_2) is μ -convex for supports. Recall that the pair (X_1, X_2) of open sets X_1, X_2 in \mathbb{R}^n satisfying (1) is called μ -convex for supports [4, Definition 3.2], [5, Definition 3.2] if for any $\nu \in \mathcal{E}'(X_2)$ we have

$$\text{dist}(\text{supp } \nu, \mathbb{R}^n \setminus X_2) = \text{dist}(\text{supp } \mu * \nu, \mathbb{R}^n \setminus X_1).$$

Here $\mu * \nu$ is the convolution of distributions μ and ν defined by

$$(\mu * \nu)(f) = \mu(\nu(f(x + y))), \quad f \in \mathcal{E}(X_1),$$

$\text{dist}(A, B) = \inf\{\|x - y\| : x \in A, y \in B\}$, $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^n .

It was proved by L. Hörmander [7, Theorem 5.4, Corollary 5.4] that if $\mu_1, \mu_2 \in \mathcal{E}'(\mathbb{R}^n)$ have disjoint singular supports and μ_1 is slowly decreasing then $\mu_1 + \mu_2$ is also slowly decreasing. Later a direct proof of this Hörmander's result was given by W. Abramczuk [8, Theorem 1]. Thus, if $\mu_1, \mu_2 \in \mathcal{E}'(\mathbb{R}^n)$ have disjoint singular supports and μ_1 defines a surjective convolution

operator on $\mathcal{E}(\mathbb{R}^n)$ then the convolution operator associated to $\mu_1 + \mu_2$ is also surjective.

Afterwards there were not many works devoted to perturbations of convolution operators in spaces of infinitely differentiable functions. Among them it is necessary to mention relatively recent work of C. Fernandez, A. Galbis and D. Jornet [9] where the behavior of the perturbed convolution operator on spaces of ultradifferentiable functions in the sense of Braun, Meise and Taylor [10] was studied. They essentially used results of J. Bonet, A. Galbis and R. Meise [11] on the range of convolution operators on spaces of non-quasianalytic ultradifferentiable functions and results of R. Braun, R. Meise, D. Vogt [12] on surjectivity of convolution operators on classes of ultradifferentiable functions.

In the present note the problem of surjectivity of perturbed convolution operators is studied in spaces of infinitely differentiable functions on convex domains of \mathbb{R}^n . The statement of the problem differ from ones in [7], [8]. It is inspired by researches of S.G. Merzlyakov [13] of perturbations of convolution operators in spaces of holomorphic functions. Namely, fix a distribution $\mu \in \mathcal{E}'(\mathbb{R}^n)$ such that its support is a convex set with non-empty interior. Let X_2 be a convex domain in \mathbb{R}^n , $X_1 = X_2 + \text{supp } \mu$. Note that in this case the pair (X_1, X_2) is μ -convex for supports. It follows from theorem on supports [6, Theorem 4.3.3] and from the fact that for arbitrary convex domain $\Omega \subset \mathbb{R}^n$ and for each compact $K \subset \Omega$ we have that $\text{dist}(K, \mathbb{R}^n \setminus \Omega) = \text{dist}(\text{ch}K, \mathbb{R}^n \setminus \Omega)$. Here $\text{ch}K$ is a convex envelope of a compact K . Assume that the convolution operator $A : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$ acting by the rule $(Af)(x) = (\mu * f)(x)$ is surjective (thus, μ is invertible). Consider a linear operator $B : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$ such that for any convex compact K_2 in X_2 there exist a convex compact subset V of the interior of support of μ (denoted by $\text{supp } \mu$) and a number $N_1 \in \mathbb{Z}_+$ such that for each positive ε that is less than the distance between K_2 and the boundary of X_2 and for each $N_2 \in \mathbb{Z}_+$ there exists a number $c = c(\varepsilon, N_2) > 0$ such that

$$\|Bf\|_{K_2^\varepsilon, N_2} \leq c\|f\|_{K_2+V, N_1}, \quad f \in \mathcal{E}(X_1). \quad (2)$$

The main result of the paper is the following

Theorem. *The operator $A + B : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$ is surjective.*

1.2. Organization of the article. In section 2 there are given two useful auxiliary results. The first is the Phragmen-Lindelöf type result (see Proposition 1). The second is the Division Theorem of L. Hörmander (see e.g., [14, Corollary 2.6.]). Also here we recall definitions of two types of locally convex spaces introduced by José Sebastião e Silva in [15]. The main result is proved in section 3. In section 4 we give an example of the operator B .

1.3. Some notations. For $u = (u_1, \dots, u_n) \in \mathbb{R}^n(\mathbb{C}^n)$, $v = (v_1, \dots, v_n) \in \mathbb{R}^n(\mathbb{C}^n)$ $\langle u, v \rangle = u_1 v_1 + \dots + u_n v_n$ and $\|u\|$ denotes the Euclidean norm in $\mathbb{R}^n(\mathbb{C}^n)$.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ $|\alpha| = \alpha_1 + \dots + \alpha_n$, D^α is the corresponding derivative.

If $\Omega \subset \mathbb{R}^n$ then $\overline{\Omega}$, $\text{int } \Omega$, $\partial\Omega$, $ch \Omega$ denote its closure, interior, boundary and convex envelope, respectively. For $\varepsilon > 0$ let $\Omega^\varepsilon = \{x \in \mathbb{R}^n : \|x - y\| \leq \varepsilon \text{ for some } y \in \Omega\}$.

For $r > 0$ let $D(r) = \{x \in \mathbb{R}^n : \|x\| < r\}$.

Supporting function H_K of convex compact set $K \subset \mathbb{R}^n$ is defined by $H_K(y) = \max_{t \in K} \langle y, t \rangle$, $y \in \mathbb{R}^n$.

$H(\mathbb{C}^n)$ is the space of entire functions on \mathbb{C}^n .

For a locally convex space E let E^* be the strong dual space.

2 Preliminaries

2.1. Auxiliary results. In the proof of the Theorem the following two results will be useful.

Proposition 1. *Let b be a non-negative convex positively homogeneous of order 1 function on \mathbb{C}^n and $g \in H(\mathbb{C}^n)$. Assume that for each $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that*

$$|g(z)| \leq c_\varepsilon \exp(b(z) + \varepsilon \|z\|), \quad z \in \mathbb{C}^n,$$

and for some $M > 0$ and $N \in \mathbb{Z}_+$

$$|g(x)| \leq M(1 + \|x\|)^N, \quad x \in \mathbb{R}^n.$$

Then

$$|g(z)| \leq 2^{\frac{N}{2}} M(1 + \|z\|)^{2N} \exp(b(i \text{Im } z)), \quad z \in \mathbb{C}^n.$$

It is an easy consequence of the Lemma below that was proved in fact in [16]. To formulate it let us introduce a space $\mathcal{P}_a(T_C)$ as follows. Let C be an open convex cone in \mathbb{R}^n with an apex at the origin and a is a nonnegative convex continuous positively homogeneous function of degree 1 on $\mathbb{R}^n + i\overline{C}$. Then $\mathcal{P}_a(T_C)$ is the space of functions f holomorphic on tube domain $T_C = \mathbb{R}^n + iC$ and satisfying the condition: for any $\varepsilon > 0$ there exists a constant $c = c_{\varepsilon, f} > 0$ such that

$$|f(z)| \leq c \exp(a(z) + \varepsilon \|z\|), \quad z \in \mathbb{R}^n + iC.$$

Lemma. Let $g \in \mathcal{P}_a(T_C)$ and for $\xi \in \mathbb{R}^n$ $\overline{\lim}_{\substack{z \rightarrow \xi, \\ z \in T_C}} |g(z)| \leq M$.

Then

$$|g(x + iy)| \leq M \exp(a(iy)), \quad x + iy \in T_C.$$

Remark. In [16, Lemma] it is assumed that C is acute. Analysis of the proof of this Lemma shows that this condition on C is unnecessary.

The following result was obtained by L. Hörmander (see e.g., [14, Corollary 2.6.]).

Proposition 2. For $j=1, 2, 3$ let $u_j \in \mathcal{E}'(\mathbb{R}^n)$, let

$$H_j(\eta) = \sup\{x, \eta\}, \quad x \in \text{supp } u_j\},$$

and let U_j be the Fourier-Laplace transform of u_j . Assume that $U_2 = \frac{U_3}{U_1}$ is entire. Then it follows that $H_2 = H_3 - H_1$ is a supporting function and that for every $\varepsilon > 0$

$$|U_2(\zeta)| \leq C_\varepsilon \exp(H_2(\text{Im} \zeta) + \varepsilon \|\zeta\|), \quad \zeta \in \mathbb{C}^n.$$

2.2. Two definitions. Recall the definitions of (M^*) -space and (LN^*) -space from [15].

Definition 1. (M^*) -space is a locally convex space F which is the projective limit of a sequence of normed spaces F_k with linear continuous mappings $g_{mk} : F_k \rightarrow F_m$, $m < k$, such that $g_{k,k+1}$ is compact for each $k \in \mathbb{N}$.

Definition 2. (LN^*) -space is a locally convex space E which is the inductive limit of an increasing sequence of normed spaces E_k such that the unit ball of E_k is relatively compact in E_{k+1} for each $k \in \mathbb{N}$, i.e. such that the inclusion map from E_k into E_{k+1} is compact.

2.3. Some additional notations and notions used in the proof of Theorem. If X is an open set in \mathbb{R}^n and $(K_m)_{m=1}^\infty$ is a sequence of compact subsets of X such that $K_m \subset \text{int } K_{m+1}$ ($m = 1, 2, \dots$) and $X = \bigcup_{k=1}^\infty K_m$ then let $C^m(K_m)$ be a normed space of functions f smooth up to the order m in K_m with a norm

$$p_m(f) = \sup_{x \in K_m, |\alpha| \leq m} |(D^\alpha f)(x)|.$$

Note that $\mathcal{E}(X)$ is a projective limit of spaces $C^m(K_m)$. Moreover, $\mathcal{E}(X)$ is dense in each $C^m(K_m)$ and embeddings $i_m : C^{m+1}(K_{m+1}) \rightarrow C^m(K_m)$ are compact. So $\mathcal{E}(X)$ is an (M^*) -space. Hence, $\mathcal{E}^*(X)$ is (LN^*) space and $\mathcal{E}^*(\Omega)$ is an inductive limit of spaces $(C^m(K_m))^*$ [15, Theorem 5].

3 Proof of the Theorem

The theorem will be proved if we show that the image of the operator $A + B$ is closed and dense in $\mathcal{E}(X_2)$.

First show that the image of $A + B$ is closed in $\mathcal{E}(X_2)$. Since $\mathcal{E}(X_1)$ and $\mathcal{E}(X_2)$ are Frechet spaces then closedness of the image of the operator $A + B$ is equivalent to closedness of the image of an adjoint operator $(A + B)^*$ [17, 8.6.13, Theorem]. Since $\mathcal{E}^*(X_1)$ is an (LN^*) -space then to show that the image of the operator $(A + B)^*$ is closed it is sufficient to prove that the image of the operator $(A + B)^*$ is sequentially closed (see [15, Proposition 8]). So let functionals $S_k \in \mathcal{E}^*(X_2)$ be such that the sequence $((A + B)^* S_k)_{k=1}^\infty$ converges to $F \in \mathcal{E}^*(X_1)$ in $\mathcal{E}^*(X_1)$.

For each $m \in \mathbb{N}$ let $X_{2,m}$ be open bounded convex subset of X_2 such that $\overline{X_{2,m}} \subset X_{2,m+1}$, $X_2 = \bigcup_{m=1}^\infty \overline{X_{2,m}}$. Put $X_{1,m} = X_{2,m} + \text{supp } \mu$. Then $\overline{X_{1,m}} \subset X_{1,m+1}$, $X_1 = \bigcup_{m=1}^\infty \overline{X_{1,m}}$.

Since $\mathcal{E}^*(X_1)$ is an (LN^*) -space then by properties of (LN^*) -spaces [15, Theorem 2, Corollary 1] there is $p \in \mathbb{N}$ such that functionals $F_k := (A + B)^* S_k$ and F belong to $(C^p(\overline{X_{1,p}}))^*$ and the sequence $(F_k)_{k=1}^\infty$ converges to F in $(C^p(\overline{X_{1,p}}))^*$. Thus, supports of functionals F_k and F are in $\overline{X_{1,p}}$ and the order of distributions F_k and F is not more than p .

Let $2r_p := \text{dist}(\overline{X_{2,p}}, \partial X_{2,p+1})$, $\tilde{X}_2 := X_{2,p} + D(r_p)$ and $\tilde{X}_1 := \tilde{X}_2 + \text{supp } \mu$. Note that \tilde{X}_1 and \tilde{X}_2 are bounded open convex sets in \mathbb{R}^n and the pair $(\tilde{X}_1, \tilde{X}_2)$ is μ -convex for supports.

Denote by \tilde{A} a convolution operator $f \in \mathcal{E}(\tilde{X}_1) \rightarrow \mu * f$. Obviously \tilde{A} is acting from $\mathcal{E}(\tilde{X}_1)$ to $\mathcal{E}(\tilde{X}_2)$ linearly and continuously and if $f \in \mathcal{E}(X_1)$ then $\tilde{A}f = \tilde{A}f$. By the (earlier cited) result of L. Hörmander [5], [6, Theorem 16.5.7] we have that $\tilde{A}(\mathcal{E}(\tilde{X}_1)) = \mathcal{E}(\tilde{X}_2)$.

Next, using the inequality (2) the operator B can be extended (uniquely) to a linear continuous operator \tilde{B} acting from $\mathcal{E}(\tilde{X}_1)$ to $\mathcal{E}(\tilde{X}_2)$. Moreover, for each convex compact $\tilde{K}_2 \subset \tilde{X}_2$ there exists a compact $V \subset \text{int}(\text{supp } \mu)$ and a number $N_1 \in \mathbb{Z}_+$ such that for each $\varepsilon \in (0, \text{dist}(\tilde{K}_2, \partial \tilde{X}_2))$ and for each $N_2 \in \mathbb{Z}_+$ there exists a number $c = c(\varepsilon, N_2) > 0$ such that

$$\|\tilde{B}f\|_{\tilde{K}_2^\varepsilon, N_2} \leq c\|f\|_{\tilde{K}_2+V, N_1}, \quad f \in \mathcal{E}(\tilde{X}_1).$$

Putting here $\tilde{K}_2 = \overline{X_{2,p}}$ we see that \tilde{B} is a compact operator from $\mathcal{E}(\tilde{X}_1)$ to $\mathcal{E}(\tilde{X}_2)$. By Theorem 9.6.7 in [17] the image of the operator $\tilde{A} + \tilde{B}$ is closed in $\mathcal{E}(\tilde{X}_2)$. Hence, the image of the operator $(\tilde{A} + \tilde{B})^*$ is closed in $\mathcal{E}^*(\tilde{X}_1)$.

For each $j \in \mathbb{N}$ let $X_{2,j} = X_{2,p} + D(\frac{j}{j+1}r_p)$. Then $\tilde{X}_2 = \bigcup_{j=1}^\infty \overline{X_{2,j}}$, $\tilde{X}_1 = \bigcup_{j=1}^\infty (\overline{X_{2,j}} + \text{supp } \mu)$. Note that for some $m \in \mathbb{N}$ supports of functionals F , $F_k = (A + B)^* S_k$ ($k = 1, 2, \dots$) are in $\overline{X_{2,m}} + \text{supp } \mu$.

Now take an arbitrary functional S_k and show that convex envelope W_k of its support is contained in $\overline{\widetilde{X}}_{2,m+2}$. Assume the contrary. Then there is a point $\xi \in W_k$ which is not in $\overline{\widetilde{X}}_{2,m+2}$. Next, there exists a hyperplane in \mathbb{R}^n dividing $\overline{\widetilde{X}}_{2,m+2}$ and ξ . So we can find a point $y_0 \in \mathbb{R}^n$ such that

$$H_{W_k}(y_0) > H_{\overline{\widetilde{X}}_{2,m+2}}(y_0). \quad (3)$$

Denote the order of distribution S_k by $N_{2,k}$. Take $\delta_1 > 0$ so small that $W_k^{\delta_1} \Subset X_2$. Then there is a constant $a_{\delta_1,k} > 0$ such that

$$|(B^*S_k)(f)| = |S_k(Bf)| \leq a_{\delta,k} \|Bf\|_{W_k^{\delta_1}, N_{2,k}}, \quad f \in \mathcal{E}(X_1).$$

By the condition on B (see the inequality (2)) there are a convex compact $V \subset \text{int}(\text{supp } \mu)$ and a number $N_1 \in \mathbb{Z}_+$ such that for a taken small $\delta_1 > 0$ there exists a constant $c_{\delta_1,k} > 0$ such that

$$|(B^*S_k)(f)| \leq c_{\delta_1,k} \|f\|_{W_k+V, N_1}, \quad f \in \mathcal{E}(X_1).$$

From this we have that for all $z \in \mathbb{C}^n$

$$|(\widehat{B^*S_k})(z)| \leq c_{\delta_1,k} (1 + \|z\|)^{N_1} \exp(H_{W_k}(\text{Im } z) + H_V(\text{Im } z)). \quad (4)$$

Taking into account that for some $d > 0$

$$H_V(x) \leq H_{\text{supp } \mu}(x) - d\|\text{Im } x\|, \quad x \in \mathbb{R}^n.$$

we get from (4) that for all $z \in \mathbb{C}^n$

$$|(\widehat{B^*S_k})(z)| \leq c_{\delta_1,k} (1 + \|z\|)^{N_1} e^{H_{W_k}(\text{Im } z) + H_{\text{supp } \mu}(\text{Im } z) - d\|\text{Im } z\|}. \quad (5)$$

Further, since $F_k \in \mathcal{E}^*(X_1)$, $\text{supp } F_k \subset \overline{\widetilde{X}}_{2,m} + \text{supp } \mu$ and the order of distribution F_k is not more than p , then for each $\delta > 0$ there exists a constant $m_{\delta,k} > 0$ such that for each $z \in \mathbb{C}^n$

$$|\hat{F}_k(z)| \leq m_{\delta,k} (1 + \|z\|)^p \exp(H_{\overline{\widetilde{X}}_{2,m}}(\text{Im } z) + H_{\text{supp } \mu}(\text{Im } z) + \delta\|\text{Im } z\|). \quad (6)$$

Using estimates (5) and (6) with $\delta = \frac{r_p}{2(m+1)(m+2)}$ we obtain that for all $z \in \mathbb{C}^n$

$$|(\widehat{A^*S_k})(z)| \leq a(1 + \|z\|)^b e^{H_{\text{ch}(W_k \cup \overline{\widetilde{X}}_{2,m+1}) + \text{supp } \mu}(\text{Im } z) - \gamma\|\text{Im } z\|}, \quad (7)$$

where $\gamma = \min(d, \delta)$, $a = \max(c_{\delta_1,k}, m_{\delta,k})$ and $b = \max(p, N_1)$.

Take a number $\gamma_1 \in (0, \gamma)$. We can find a convex compact $\Omega_k \subset \text{int} (ch(W_k \cup \overline{\tilde{X}}_{2,m+1}))$ such that

$$H_{ch(W_k \cup \overline{\tilde{X}}_{2,m+1})}(y) - H_{\Omega_k}(y) \leq \gamma_1 \|y\|, \quad y \in \mathbb{R}^n.$$

Then from (7) we have that

$$|\widehat{(A^*S_k)}(z)| \leq a(1 + \|z\|)^b e^{H_{\Omega_k + \text{supp } \mu}(Im z)},$$

Note that by the Paley-Wiener-Schwartz theorem [6, Theorem 7.3.1] this means that the support of A^*S_k is contained in $\Omega_k + \text{supp } \mu$.

Now taking into account the equality

$$\widehat{(A^*S_k)}(z) = \hat{S}_k(z)\hat{\mu}(z), \quad z \in \mathbb{C}^n,$$

and Proposition 2 we get that $H_{\text{supp } (A^*S_k)}(x) - H_{\text{supp } \mu}$ is a supporting function of some convex compact $G_k \subset \mathbb{R}^n$ and for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|\hat{S}_k(z)| \leq C_\varepsilon \exp(H_{G_k}(Im z) + \varepsilon \|z\|), \quad z \in \mathbb{C}^n. \quad (8)$$

Also by the Paley-Wiener-Schwartz theorem [6, Theorem 7.3.1] for some $M_k > 0$ we have that

$$|\hat{S}_k(x)| \leq M_k(1 + \|x\|)^{N_{2,k}}, \quad x \in \mathbb{R}^n,$$

From this inequality and inequality (8) using Proposition 1 we get that

$$|\hat{S}_k(z)| \leq M_k(1 + \|z\|)^{2N_{2,k}} e^{H_{G_k}(Im z)}, \quad z \in \mathbb{C}^n.$$

Again using the Paley-Wiener-Schwartz theorem [6, Theorem 7.3.1] we obtain that the support of S_k is contained in G_k . Therefore, for all $y \in \mathbb{R}^n$ we have that

$$\begin{aligned} H_{W_k}(y) &\leq H_{G_k}(y) = H_{\text{supp } (A^*S_k)}(y) - H_{\text{supp } \mu}(y) \leq \\ &\leq H_{\Omega_k + \text{supp } \mu}(y) - H_{\text{supp } \mu}(y) = H_{\Omega_k}(y). \end{aligned}$$

From this taking into account that $\Omega_k \subset \text{int} (ch(W_k \cup \overline{\tilde{X}}_{2,m+1}))$ we get that

$$H_{W_k}(y) < \max(H_{W_k}(y), H_{\overline{\tilde{X}}_{2,m+1}}(y)), \quad y \in \mathbb{R}^n.$$

But it is impossible in view of (3). Thus, for each $k \in \mathbb{N}$ convex envelope W_k of support of functional S_k ($k = 1, 2, \dots$) is contained in $\overline{\tilde{X}}_{2,m+2}$.

Now let $\eta \in \mathcal{E}(\mathbb{R}^n)$ be a function with a support in $\overline{\tilde{X}_{2,m+4}}$ such that $0 \leq \eta(x) \leq 1$ for all $x \in \mathbb{R}^n$ and $\eta(x) = 1$ for $x \in \overline{\tilde{X}_{2,m+3}}$. For each $k \in \mathbb{N}$ define a functional \tilde{S}_k on $\mathcal{E}(\tilde{X}_2)$ by the rule: $\tilde{S}_k(f) = S_k(\eta f)$, $f \in \mathcal{E}(\tilde{X}_2)$. Obviously, $\tilde{S}_k \in \mathcal{E}^*(\tilde{X}_2)$ and $\tilde{S}_k(f) = S_k(f)$, $f \in \mathcal{E}(X_2)$. Note that since for each $f \in \mathcal{E}(X_1)$ $(A+B)(f) = (\tilde{A}+\tilde{B})(f)$ then functionals $(A+B)^*S_k$ and $(\tilde{A}+\tilde{B})^*\tilde{S}_k$ ($k = 1, 2, \dots$) coincide on $\mathcal{E}(X_1)$. Now taking into account that $\mathcal{E}(X_1)$ is dense in $\mathcal{E}(\tilde{X}_1)$ we get that $(\tilde{A}+\tilde{B})^*\tilde{S}_k$ is the (unique) extension of the functional $(A+B)^*S_k$ to $\mathcal{E}(\tilde{X}_1)$.

Show that functionals $(\tilde{A}+\tilde{B})^*\tilde{S}_k$ converge in $\mathcal{E}^*(\tilde{X}_1)$. First note that the sequence $((\tilde{A}+\tilde{B})^*\tilde{S}_k)_{k=1}^\infty$ is fundamental in $\mathcal{E}^*(\tilde{X}_1)$. Indeed, let \mathcal{B} be an arbitrary bounded set in $\mathcal{E}(\tilde{X}_1)$ and

$$\mathcal{B}^\circ = \{F \in \mathcal{E}^*(\tilde{X}_1) : |F(f)| \leq 1 \ \forall f \in \mathcal{B}\}$$

its polar set. Take a function $\omega \in \mathcal{E}(\mathbb{R}^n)$ with a support in $\overline{\tilde{X}_{2,m+4}} + \text{supp } \mu$ such that $0 \leq \omega(x) \leq 1$ for all $x \in \mathbb{R}^n$ and $\omega(x) = 1$ for $x \in \overline{\tilde{X}_{2,m+3}} + \text{supp } \mu$. Since the support of functionals \tilde{S}_k is in $\overline{\tilde{X}_{2,m+2}}$ then the support of functionals $(\tilde{A}+\tilde{B})^*\tilde{S}_k$ is contained in $\tilde{X}_{2,m+2} + \text{supp } \mu$. So for each $f \in \mathcal{E}(\tilde{X}_1)$ and all $k, m \in \mathbb{N}$ we have that

$$((\tilde{A}+\tilde{B})^*\tilde{S}_k)(f) - ((\tilde{A}+\tilde{B})^*\tilde{S}_m)(f) = ((\tilde{A}+\tilde{B})^*\tilde{S}_k)(\omega f) - ((\tilde{A}+\tilde{B})^*\tilde{S}_m)(\omega f).$$

We may consider ωf as an element of $\mathcal{E}(X_1)$ setting $(\omega f)(x) = 0$ for $x \in X_1 \setminus (\overline{\tilde{X}_{2,m+4}} + \text{supp } \mu)$. Then

$$((\tilde{A}+\tilde{B})^*\tilde{S}_k)(f) - ((\tilde{A}+\tilde{B})^*\tilde{S}_m)(f) = ((A+B)^*S_k)(\omega f) - ((A+B)^*S_m)(\omega f).$$

Note that the set $\omega\mathcal{B} = \{\omega f : f \in \mathcal{B}\}$ is bounded in $\mathcal{E}(X_1)$. Since the sequence $((A+B)^*S_k)_{k=1}^\infty$ is converging in $\mathcal{E}^*(X_1)$ then it is fundamental in $\mathcal{E}^*(X_1)$. So there is $N \in \mathbb{N}$ such that for all natural numbers k, m : $k, m \geq N$ and $g \in \omega\mathcal{B}$ we have $|((A+B)^*S_k)(g) - ((A+B)^*S_m)(g)| \leq 1$. Hence, for all natural k, m : $k, m \geq N$ and $f \in \mathcal{B}$ we get that

$$|((\tilde{A}+\tilde{B})^*\tilde{S}_k)(f) - ((\tilde{A}+\tilde{B})^*\tilde{S}_m)(f)| \leq 1.$$

This means that for all natural k, m : $k, m \geq N$ and $f \in \mathcal{B}$ we have that $(\tilde{A}+\tilde{B})^*\tilde{S}_k - (\tilde{A}+\tilde{B})^*\tilde{S}_m \in \mathcal{B}^\circ$. Thus, we have proved that the sequence $((\tilde{A}+\tilde{B})^*\tilde{S}_k)_{k=1}^\infty$ is fundamental in $\mathcal{E}^*(\tilde{X}_1)$. Finally, since $\mathcal{E}^*(\tilde{X}_1)$ is complete then we get that the sequence $((\tilde{A}+\tilde{B})^*\tilde{S}_k)_{k=1}^\infty$ is converging in $\mathcal{E}^*(\tilde{X}_1)$ to some element $\tilde{T} \in \mathcal{E}^*(\tilde{X}_1)$. But $(\tilde{A}+\tilde{B})^*(\mathcal{E}^*(\tilde{X}_2))$ is closed in $\mathcal{E}^*(\tilde{X}_1)$. Hence, there exists a functional $\tilde{S} \in \mathcal{E}^*(\tilde{X}_2)$ such that $\tilde{T} = (\tilde{A}+\tilde{B})^*\tilde{S}$. Let S be restriction of \tilde{S} on $\mathcal{E}(X_2)$. Then for each $f \in \mathcal{E}(X_1)$ we have that $\tilde{T}(f) = T(f)$. Indeed,

$$\tilde{T}(f) = \lim_{k \rightarrow \infty} ((\tilde{A}+\tilde{B})^*(\tilde{S}_k))(f) = \lim_{k \rightarrow \infty} \tilde{S}_k((\tilde{A}+\tilde{B})f) = \lim_{k \rightarrow \infty} \tilde{S}_k((A+B)f) =$$

$$= \lim_{k \rightarrow \infty} S_k((A+B)f) = \lim_{k \rightarrow \infty} ((A+B)^* S_k)(f) = T(f).$$

From this and the following chain of equalities

$$\begin{aligned} \tilde{T}(f) &= \lim_{k \rightarrow \infty} ((\tilde{A} + \tilde{B})^*(\tilde{S}_k))(f) = ((\tilde{A} + \tilde{B})^*(\tilde{S}))(f) = \tilde{S}((\tilde{A} + \tilde{B})f) = \\ &= \tilde{S}((A+B)f) = S((A+B)f) = ((A+B)^* S)(f) \end{aligned}$$

it follows that $T = (A+B)^* S$. Thus, the image of the operator $(A+B)^*$ is closed in $\mathcal{E}^*(X_1)$. Consequently, the image of the operator $A+B$ is closed in $\mathcal{E}(X_2)$.

Now we prove that the image of the operator $A+B$ is dense in $\mathcal{E}(X_2)$. It will be done if we show that an arbitrary functional $S \in \mathcal{E}^*(X_2)$ with the property $S((A+B)f) = 0$ for all $f \in \mathcal{E}(X_1)$ is a zero functional. Assume the contrary. Then the support of S is not empty. Let N be the order of distribution S and $\delta > 0$ be so small that $(\text{supp } S)^\delta \subseteq X_2$. Then there exists a constant $c_\delta > 0$ such that

$$|S(g)| \leq c_\delta \|g\|_{(\text{supp } S)^\delta, N}, \quad g \in \mathcal{E}(X_2).$$

From this and the inequality (2) it follows that there exists a convex compact $V \subset \text{int}(\text{supp } \mu)$, a number $N_1 \in \mathbb{Z}_+$ (depending on $\text{ch}(\text{supp } S)$) and a constant $C_\delta > 0$ such that for each $f \in \mathcal{E}(X_1)$

$$|(B^* S)(f)| \leq C_\delta \|f\|_{\text{ch}(\text{supp } S) + V, N_1}.$$

Hence, the support of functional $B^* S$ is contained in $\text{ch}(\text{supp } S) + V$. From the other hand from the equality $B^* S = -\mu * S$ and by the theorem on supports [6, Theorem 4.3.3] we have that $\text{ch}(\text{supp } B^* S) = \text{ch}(\text{supp } S) + \text{supp } \mu$. Thus, $\text{ch}(\text{supp } S) + \text{supp } \mu \subset \text{ch}(\text{supp } S) + V$. But this inclusion is impossible since convex compact V is contained in the interior of the support of μ . Hence, our assumption that S is not a zero functional was false. Thus, $S = 0$. This means that the image of the operator $A+B$ is dense in $\mathcal{E}(X_2)$.

Theorem is proved.

4 Example of the operator B

Let $\mu \in \mathcal{E}'(\mathbb{R}^n)$ be an invertible distribution and $\text{supp } \mu = \overline{D(1)}$. Distributions with these properties can be constructed (see, e.g., [8, Theorem 1, Theorem 3, Theorem 4]). Let $X_2 = D(1)$, $X_1 = D(2)$. Let $A : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$ be a convolution operator acting by the rule $(Af)(x) = (\mu * f)(x)$, $x \in X_1$. Take a function $b \in \mathcal{E}(\mathbb{R}^{2n})$ with the support in $\overline{D(\frac{1}{4})} \times \overline{D(\frac{1}{4})}$. Define the operator $B : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$ acting by the rule

$$(Bf)(x) = \int_{\mathbb{R}^n} b(x, \xi) f(x + \xi) d\xi, \quad \|x\| \leq \frac{1}{4},$$

$$(Bf)(x) = 0, \quad \frac{1}{4} < \|x\| < 1.$$

Let K be a convex compact in X_2 and $\gamma := \text{dist}(K, \partial X_2)$. Show that there exists a convex compact $V \subset \text{int}(\text{supp } \mu)$ such that for any $\varepsilon \in (0, \gamma)$ and for any $N_2 \in \mathbb{Z}_+$ there exists a constant $c = c(\varepsilon, N_2) > 0$ such that

$$\|Bf\|_{K^\varepsilon, N_2} \leq c \|f\|_{K+V, 0}, \quad f \in \mathcal{E}(X_1).$$

Obviously, for each $\varepsilon \in (0, \gamma)$ and for each $N_2 \in \mathbb{Z}_+$ there exists a constant $C > 0$ depending on b and N_2 such that for each $f \in \mathcal{E}(X_1)$ we have that

$$\|Bf\|_{K^\varepsilon, N_2} = \|Bf\|_{K^\varepsilon \cap \overline{D(\frac{1}{4})}, N_2} \leq C_1 \|f\|_{(K^\varepsilon \cap \overline{D(\frac{1}{4})}) + \overline{D(\frac{1}{4})}, 0}. \quad (9)$$

If $\gamma \in (0, \frac{3}{4})$ then from (9) we have that

$$\|Bf\|_{K^\varepsilon, N_2} \leq C_1 \|f\|_{K^\gamma + \overline{D(\frac{1}{4})}, 0} = C_1 \|f\|_{K + \overline{D(\gamma + \frac{1}{4})}, 0}$$

Hence, in this case we can put $V = \overline{D(\gamma + \frac{1}{4})}$. If $\gamma \in [\frac{3}{4}, 1]$ then $K \subset \overline{D(\frac{1}{4})}$ and from (9) we have that

$$\|Bf\|_{K^\varepsilon, N_2} \leq C_1 \|f\|_{\overline{D(\frac{1}{2})}, 0} \leq C_1 \|f\|_{K + \overline{D(\frac{3}{4})}, 0}.$$

So if $\gamma \in [\frac{3}{4}, 1]$ then we can put $V = \overline{D(\frac{3}{4})}$.

Thus, by Theorem the operator $A + B : \mathcal{E}(X_1) \rightarrow \mathcal{E}(X_2)$ is surjective.

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